Derivation of the equations for the X-ray transform

The γ-radiation is attenuated exponentially along its passage through the material, because of scattering and absorption interactions with the atoms in the material:

$$I = I_o e^{-\int_0^t f(t)dt} \tag{1}$$

Here t indicates the thickness of the attenuating material that has been penetrated and I_o is the incident radiation intensity. The attenuation properties of the material are described by the product f(t) = μ(t) ρ(t), where μ(t) is the mass-attenuation coefficient (m^2/kg) and ρ(t) is the density (kg/m^3).

The fan beam projections g(σ,β) of an object f(x,y) are line integrals obtained by the **X-ray transform**:

$$g(\sigma,\beta) = \iint f(x,y)\delta\left[D\sin\sigma - x\cos(\sigma+\beta) - y\sin(\sigma+\beta)\right]dxdy \tag{2}$$

$$= \int_{t_{min}(\sigma,\beta)}^{t_{max}(\sigma,\beta)} f\left[-D\sin\beta + t\sin(\sigma+\beta)\ ,D\cos\beta - t\cos(\sigma+\beta)\right]dt \tag{3}$$

where the notation is illustrated in Fig. 1. Each line integral, g(σ,β), is defined by the view angle of the projection, $\beta \in [0, 2\pi]$, which expresses the location of the x-ray source, and the fan angle, $\sigma \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which selects a particular ray in the projection. The radius of motion of the x-ray source is D.

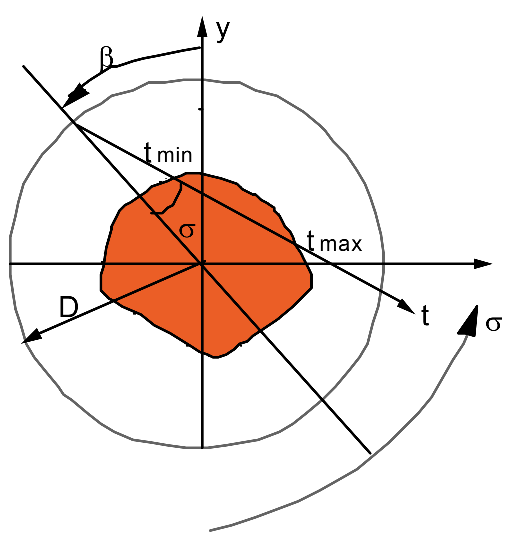

FIG. 1
The geometry of the X-ray transform.

In the **backprojection-of-filtered-projections** reconstruction algorithm, the image f(x,y) is obtained from the integral:

$$f(x,y)=\frac{1}{2}\int_0^{2\pi}\frac{\tilde{g}(\sigma,\beta)\big|_{\sigma=\sigma'(x,y,\beta)}}{L^2(x,y,\beta)}\,d\beta \tag{4}$$

The integral over β is known as **the backprojection** and $\tilde{g}(\sigma,\beta)$ are **the filtered projections** given by:

$$\tilde{g}(\sigma',\beta)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(\sigma,\beta)\,h(\sin(\sigma'-\sigma))\,D\cos\sigma\,d\sigma$$

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where:

$$h(\sin(\sigma'-\sigma))=\int_{-\infty}^{\infty}|\omega_\ell|\,e^{-i\omega_\ell\sin(\sigma'-\sigma)}d\omega_\ell$$

$$=\left(\frac{\sigma'-\sigma}{\sin(\sigma'-\sigma)}\right)^2\int_{-\infty}^{\infty}|\omega_\sigma|\,e^{-\omega_\sigma(\sigma'-\sigma)}d\omega_\sigma=\left(\frac{\sigma'-\sigma}{\sin(\sigma'-\sigma)}\right)^2 h(\sigma'-\sigma)$$

and $\sigma'(x,y,\beta)$ and $L(x,y,\beta)$ describe the fan angle of the ray traversing through the point (x,y), and the distance from the x-ray source to the point (x,y) in the image is:

$$\sigma'(x,y,\beta)=\arctan\left[\frac{x\cos\beta+y\sin\beta}{D+x\sin\beta-y\cos\beta}\right] \tag{5}$$

$$L(x,y,\beta)=\sqrt{(x\cos\beta+y\sin\beta)^2+(D+x\sin\beta-y\cos\beta)^2} \tag{6}$$

Suggested reading:

S.R. Deans, *The Radon Transform and Some of Its Applications*, John Wiley & Sons, Inc. (1983).

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