Finite Elements for Crack Problems

The concept of finite elements was introduced in the last section for determining temperature distribution in mass concrete. Next we will discuss how finite elements can be used to determine stress intensity factors K_I and K_{II} for complex geometries.

As mentioned previously, the finite element method is a powerful tool for the numerical treatment of partial differential equations. For elasticity problems with complex geometric boundaries it is usually impossible to find an exact solution for the displacements and stresses. In order to construct an approximate solution, the domain under consideration is divided into subdomains called finite elements. For every finite element, linearly independent basis functions similar to those used for the heat transfer problems can be used in order to approximate the displacement field. Restricting our attention to plane problems, the nodal j^{th} values of a finite element are usually chosen to be the displacement components u_j and v_j (see Fig. 1).

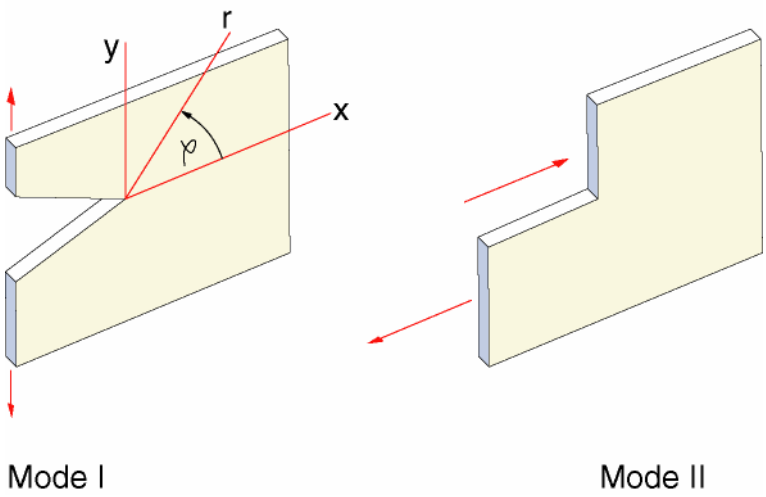

Figure 1 Coordinate system at the tip of the crack (From R. Piltner, *Spezielle finite element emit Lochen, Ecken und Russen, unter analystischen Teillosungen*, VDI-Verlag, Dusseldorf, n. 96, 1982)

The basis functions for standard displacement finite elements consist of shape functions multiplied by the unknown nodal values. By coupling the finite elements, we "glue" the pieces of our solution together. The unknowns in the finite element solution are the nodal values. For the unknown nodal values we require that the potential energy of the system is minimized, thereby setting up the system of equations for the unknowns.

The stresses in standard displacement elements are finite. Therefore, standard displacement elements are not appropriate to approximate the stress singularities at the

crack tip. For crack problems, special finite elements are needed that include crack tip singularities in the trial functions. In addition, we would like to couple crack elements with standard displacement elements for which polynomials are used as approximation functions.

In order to couple crack elements with standard displacement elements, it is critical that the displacements along the edges of adjacent elements are compatible. The procedure of coupling a crack element with a standard displacement element is illustrated in Fig. 2.

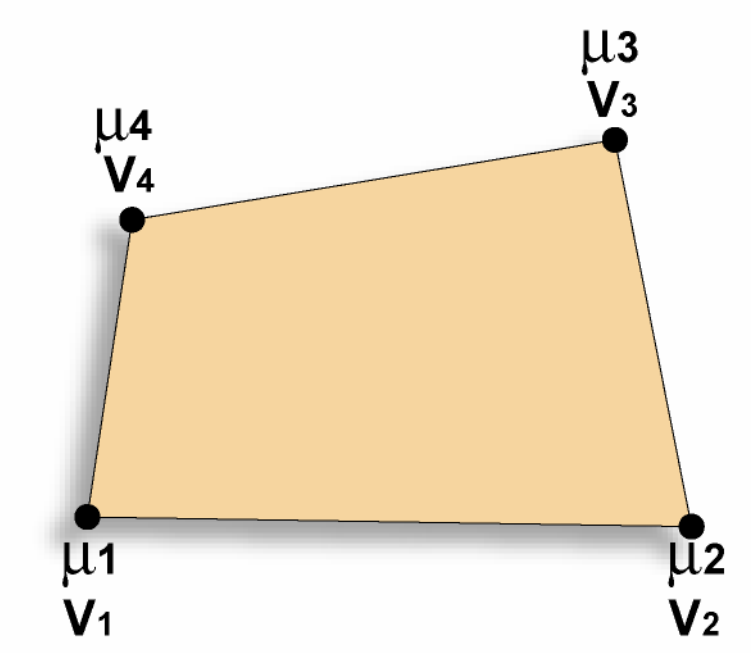

Figure 2 Quadrilateral 4-node element

The crack element requires displacement trial functions with the following properties:

The linearly independent trial functions satisfy the equilibrium equations;

The trial functions satisfy the stress free boundary conditions on the crack surface; and

The trial functions for the displacements contain terms are proportional to $\sqrt{r}$ *so that the associated stresses are proportional to* $1/\sqrt{r}$.

These conditions ensure that the correct form of the singular stress function terms will be used in the finite element approach. From the finite element analysis we get the coefficients of the singular stress functions for the crack tip . Apart from a factor , the coefficients of the singular stress functions are the stress intensity factors.

Two questions remain: (i) How do we systematically construct linearly independent trial functions for the displacements and stresses with the properties listed above; (ii) How can we compute a stiffness matrix for a crack element when the stresses are singular at one point.

For the construction of linearly independent trial functions, a representation of the displacements and stresses in terms of arbitrary functions is helpful. Using two complex functions, $\phi(z)$ and $\Psi(z)$, the displacements and stresses can be written in the form[1] :

$$
\begin{aligned}
2\,\mu\, u &= \mathrm{Re}\left[\kappa\,\phi(z) - z\,\overline{\phi'(z)} - \overline{\Psi(z)}\right] \\
2\,\mu\, v &= \mathrm{Im}\left[\kappa\,\phi(z) - z\,\overline{\phi'(z)} - \overline{\Psi(z)}\right] \\
\sigma_x &= \mathrm{Re}\left[2\,\phi'(z) - \bar{z}\,\phi''(z) - \Psi'(z)\right] \\
\sigma_y &= \mathrm{Re}\left[2\,\phi'(z) + \bar{z}\,\phi''(z) + \Psi'(z)\right] \\
\tau_{xy} &= \mathrm{Im}\left[\bar{z}\,\phi''(z) + \Psi'(z)\right]
\end{aligned}
\qquad (1)
$$

where $z=x+iy$, ϕ' denotes differentiation with respect to z, $2\mu = E/(1+\nu)$ and $\kappa = (3-4\nu)$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress. The advantage of using Eq. (1) is that for any choice of functions ϕ and Ψ, the equilibrium equations are automatically satisfied. For our crack element we need functions ϕ and Ψ, which ensures the satisfaction of the stress-free boundary conditions on the crack surfaces. These functions may be represented in the form of a power series as:

$$
\phi(z) = \sum_{j=0}^{N} a_j\, \varsigma^j \qquad (2)
$$

and,

$$
\Psi(z) = -\sum_{j=0}^{N} \left[\, \bar{a}_j\,(-1)^j + \frac{j}{2}\, a_j \right] \varsigma^j \qquad (3)
$$

where $a_j = \alpha_j + i\,\beta_j$ and $\varsigma = \sqrt{z}$.

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[1] N.I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff, Groningen, Holland, 1953.

Substituting Eqs. (2, 3) into (1) gives us the linearly independent, real trial functions for our crack element. Note that the terms with the index j=1 give the singular stress terms for Mode I and Mode II. For example, for σ_x we obtain the singular terms in the form:

$$\sigma_x = \frac{1}{\sqrt{r}} \cos\frac{\varphi}{2}\left[1 - \sin\frac{\varphi}{2}\sin\frac{3\varphi}{2}\right]\alpha_1 + \frac{1}{\sqrt{r}}\sin\frac{\varphi}{2}\left[2 + \cos\frac{\varphi}{2}\cos\frac{3\varphi}{2}\right]\beta_1 \tag{4}$$

Stress intensity factors K_I and K_{II} can be calculated from

$$K_I = \sqrt{2\pi}\,\alpha_1 \tag{5}$$

$$K_{II} = -\sqrt{2\pi}\,\beta_1 \tag{6}$$

Collecting unknown coefficients α_j and β_j into a vector $\underline{c}$, the displacements for our crack element can be written in matrix notation as

$$\underline{u} = \underline{U}\,\underline{c} + \underline{u}_p = \underline{u}_h + \underline{u}_p \tag{7}$$

where $\underline{u}_p$ is a particular solution involving no unknown coefficients.

If we want to take nonhomogeneous stress boundary conditions on a crack surface into account (for example, constant pressure on the crack), a particular solution can be used. Only the homogeneous solution $\underline{u}_h$ involves unknown coefficients. Since the unknowns in vector $\underline{c}$ are not associated with finite element nodal values, it is necessary to relate in some manner vector $\underline{c}$ to the vector of nodal displacements $\underline{q}$; vector $\underline{q}$ contains the nodal values $\underline{u}_j$, $\underline{v}_j$ of the chosen element nodes. In Fig. 2, a linear variation of the boundary displacements, $[\tilde{u}\ \tilde{v}]^T = \tilde{\underline{u}}$, is assumed between nodes i and j. If we want to couple the crack element with linear standard displacement elements, $\tilde{u}$, $\tilde{v}$ are chosen linear between two nodes. If the crack element is to be coupled with quadratic standard elements, a quadratic variation of the boundary displacement $\tilde{\underline{u}}$ of the crack element is chosen.

The first step for evaluating a crack element stiffness matrix, vector $\underline{c}$ of the displacement field $\underline{u}$ for the domain V^i of the crack elements is calculated such that an optimal agreement between $\underline{u}$ and $\tilde{\underline{u}}$ is achieved along the boundary of the crack element. This gives us the following relationship:

$$\underline{c} = G\,\underline{q} + \underline{g} \tag{8}$$

so that the unknowns $\underline{c}$ can be eliminated, and only the nodal displacements $\underline{q}$ will remain as unknowns of the crack element.

To evaluate the crack element stiffness matrix, the following displacement functional is used:

$$\Pi_H^i = \Pi^i + \int_{s_u^i} \underline{T}^T (\tilde{\underline{u}} - \underline{u})\, dS^i \tag{9}$$

where

$$\Pi^i = \int_{V_i} \left[1/2\ (\underline{u}^T \underline{D}^T)\, \underline{E}\, (\underline{D}\, \underline{u}) - \underline{u}^T \bar{\underline{f}} \right] dV^i - \int_{s_u^i} \underline{u}^T \overline{\underline{T}}\, dS^i \tag{10}$$

and V_i denotes the domain of the finite element, S_i is the boundary of the element, and $\underline{T}$ are the tractions along the element boundary. Using the decomposition of the displacements and tractions in the form $\underline{u} = \underline{u}_h + \underline{u}_p$ and $\underline{T} = \underline{T}_h + \underline{T}_p$, we can simplify the variational formulation. Since the displacement field for the crack element is constructed such that the governing partial differential equations (Navier-equation in matrix notation):

$$\underline{D}^T\, \underline{E}\, \underline{D}\, \underline{u} = -\bar{\underline{f}} \qquad \text{in } V \tag{11}$$

are satisfied *a priori*, Eq. (10), can be simplified to an expression with boundary integrals:

$$\Pi^i = \int_{s_u^i} 1/2\ \underline{u}_h^T \underline{T}_h\, dS^i + \int_{s_u^i} \underline{u}_h^T \underline{T}_p\, dS^i - \int_{s_u^i} \underline{u}_h^T \overline{\underline{T}}\, dS^i + \text{terms without } \underline{u}_h \text{ and } \underline{T}_h \tag{12}$$

Using the stiffness matrix of a crack element can be obtained by evaluations of the boundary integrals along the element boundary[2, 3] .It is important that the boundary conditions on the crack surface are satisfied *a priori* so that all integrals along the crack surface vanish. Therefore, we do not need to evaluate stresses at the crack tip during the calculations of the stiffness matrix, and although the stress singularities are included in the model, the evaluation of the stiffness coefficients will not be "polluted" from the presence of the singularities.

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[2] R. Piltner, *Int. J. Numer. Methods Eng.*, 21, 1471, 1985

[3] R. Piltner, in "*Local Effects in the Analysis of Structures*," Elsevier, Amsterdam, 299, 1985.